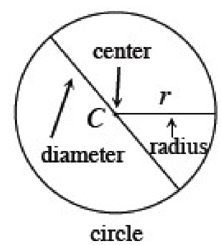
Methods and Meanings

MATH NOTES

Parts of a Circle

A **circle** is the set of all points on a flat surface that are the same distance from a fixed central point, C, referred to as its **center**. This text will use the notation $\odot C$ to name a circle with center at point C.



The **radius** is a line segment from the center to a point on the circle. Its length is usually denoted r. However, a line segment drawn through the center of the circle with both endpoints on the circle is called a **diameter** and its length is usually denoted d.

Notice that a diameter of a circle is always twice as long as the radius.

MATH NOTES

METHODS AND MEANINGS

Congruent Triangles → Congruent Corresponding Parts

As you learned in Chapter 3, if two shapes are congruent, then they have exactly the same shape and the same size. This means that if you know two triangles are congruent, you can state that corresponding parts are congruent. This can be also stated with the arrow diagram:

$$\cong \Delta\text{s} \rightarrow \cong \text{parts}$$

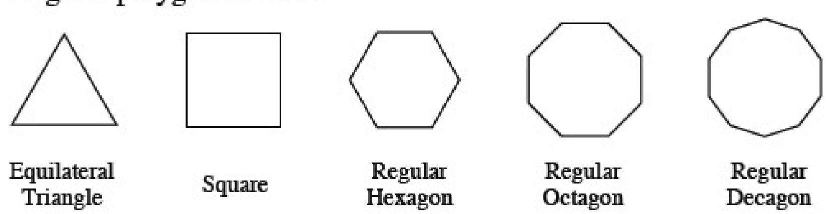
For example, if $\Delta ABC \cong \Delta PQR$, then it follows that $\angle A \cong \angle P$, $\angle B \cong \angle Q$, and $\angle C \cong \angle R$. Also, $\overline{AB} \cong \overline{PQ}$, $\overline{AC} \cong \overline{PR}$, and $\overline{BC} \cong \overline{QR}$.

MATH NOTES

METHODS AND MEANINGS

Regular Polygons

A polygon is **regular** if all its sides are congruent and its angles have equal measure. An equilateral triangle and a square are each regular polygons since they are both *equilateral* and *equiangular*. See the diagrams of common regular polygons below.



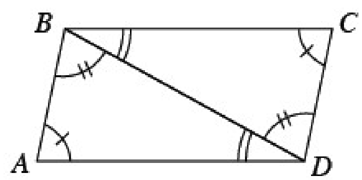
METHODS AND MEANINGS

MATH NOTES

Reflexive Property of Equality

In this lesson, you used the fact that two triangles formed by the diagonal of a parallelogram share a side of the same length to help show that the triangles were congruent.

The **Reflexive Property of Equality** states that the measure of any side or angle is equal to itself. For example, in the parallelogram at right, $\overline{BD} \cong \overline{DB}$ because of the Reflexive Property.



METHODS AND MEANINGS

MATH NOTES

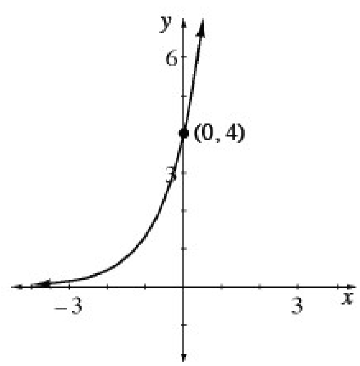
Exponential Functions

An **exponential function** has the general form $y = a \cdot b^x$, where a is the **initial value** (the y-intercept) and b is the **multiplier** (the growth). Be careful: The independent variable x has to be in the exponent. For example, $y = x^2$ is *not* an exponential equation, even though it has an exponent.

For example, in the multiple representations below, the y-intercept is $(0, 4)$ and the growth factor is 3 because the y-value is increasing by multiplying by 3.

$y = 4 \cdot 3^x$

x	y	
–3	$\frac{4}{3^3}$ or $\frac{4}{27}$	
–2	$\frac{4}{3^2}$ or $\frac{4}{9}$	
–1	$\frac{4}{3}$	
0	4	
1	12	$\times 3$
2	36	$\times 3$
3	108	



To increase or decrease a quantity by a percentage, use the multiplier for that percentage. For example, the multiplier for an increase of 7% is 100% + 7% = 1.07. The multiplier for a decrease of 7% is 100% – 7% = 0.93.

MATH NOTES

METHODS AND MEANINGS

Definitions of Quadrilaterals

When proving properties of shapes, it is necessary to know exactly how a shape is defined. Below are the definitions of several quadrilaterals that you developed in Lesson 1.3.2 and that you will need to refer to in this chapter and the chapters that follow.

Quadrilateral: A closed four-sided polygon.

Kite: A quadrilateral with two distinct pairs of consecutive congruent sides.

Trapezoid: A quadrilateral with at least one pair of parallel sides.

Parallelogram: A quadrilateral with two pairs of parallel sides.

Rhombus: A quadrilateral with four sides of equal length.

Rectangle: A quadrilateral with four right angles.

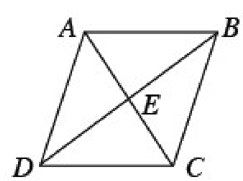
Square: A quadrilateral with four sides of equal length and four right angles.

METHODS AND MEANINGS

MATH NOTES

Diagonals of a Rhombus

A **rhombus** is defined as a quadrilateral with four sides of equal length. In addition, you proved in problem 7-62 that the diagonals of a rhombus are perpendicular bisectors of each other.



For example, in the rhombus at right, E is a midpoint of both $\overline{AC}$ and $\overline{DB}$. Therefore, $AE = CE$ and $DE = BE$. Also, $m\angle AEB = m\angle BEC = m\angle CED = m\angle DEA = 90^\circ$.

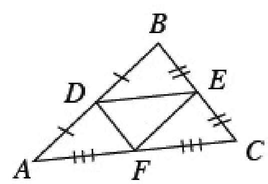
In addition, you proved in problem 7-82 that the diagonals bisect the angles of the rhombus. For example, in the diagram above, $m\angle DAE = m\angle BAE$.

METHODS AND MEANINGS

MATH NOTES

Triangle Midsegment Theorem

A **midsegment** of a triangle is a segment that connects the midpoints of any two sides of a triangle. Every triangle has three midsegments, as shown at right.



A midsegment between two sides of a triangle is half the length of and parallel to the third side of the triangle. For example, in ΔABC at right, $\overline{DE}$ is a midsegment, $\overline{DE} \parallel \overline{AC}$, and $DE = \frac{1}{2}AC$.

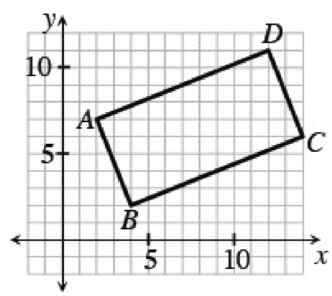
METHODS AND MEANINGS

MATH NOTES

Coordinate Geometry

Coordinate geometry is the study of geometry on a coordinate grid. Using common algebraic and geometric tools, you can learn more about a shape, such as, "*Does it have a right angle?*" or "*Are there two sides with the same length?*"

One useful tool is the Pythagorean Theorem. For example, the Pythagorean Theorem could be used to determine the length of side $\overline{AB}$ of $ABCD$ at right. By drawing the slope triangle between points A and B, the length of $\overline{AB}$ can be found to be $\sqrt{2^2+5^2}=\sqrt{29}$ units.



Similarly, slope can help analyze the relationships between the sides of a shape. If the slopes of two sides of a shape are equal, then those sides are **parallel**. For example, since the slope of $\overline{BC}=\frac{2}{5}$ and the slope of $\overline{AD}=\frac{2}{5}$, then $\overline{BC} \parallel \overline{AD}$.

Also, if the slopes of two sides of a shape are opposite reciprocals, then the sides are **perpendicular** (meaning they form a 90° angle). For example, since the slope of $\overline{BC}=\frac{2}{5}$ and the slope of $\overline{AB}=-\frac{5}{2}$, then $\overline{BC} \perp \overline{AB}$.

By using multiple algebraic and geometric tools, you can identify shapes. For example, further analysis of the sides and angles of $ABCD$ above shows that $AB=DC$ and $BC=AD$. Furthermore, all four angles measure 90°. These facts together indicate that $ABCD$ must be a rectangle.

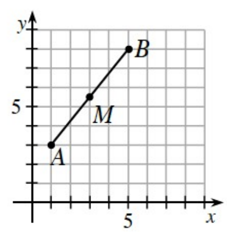
MATH NOTES-Chapters 7 - 12

MATH NOTES

METHODS AND MEANINGS

Finding a Midpoint

A **midpoint** is a point that divides a line segment into two parts of equal length. For example, M is the midpoint of $\overline{AB}$ at right.



There are several ways to find the midpoint of a line segment if the coordinates of the endpoints are known. One way is to add half the change in x ($\frac{1}{2}\Delta x$) and half of the change in y ($\frac{1}{2}\Delta y$) to the x- and y-coordinates of the starting point, respectively.

Thus, if $A(1,3)$ and $B(5,8)$, then $\Delta x = 5-1=4$ and $\Delta y = 8-3=5$. Then the x-coordinate of M is $1+\frac{1}{2}(4)=3$ and the y-coordinate is $3+\frac{1}{2}(5)=5.5$. So point M is at $(3, 5.5)$.

This strategy can be used to find other points between A and B that are a proportion of the way from a starting point. For example, if you wanted to find a point $\frac{4}{5}$ of the way from point A to point B, then this could be found by adding $\frac{4}{5}$ of Δx to the x-coordinate of point A and adding $\frac{4}{5}$ of Δy to the y-coordinate of point A. This would be the point $((1+\frac{4}{5}(4), 3+\frac{4}{5}(5))$ which is $(4.2, 7)$. Generally, a point a ratio r from $A(x_0, y_0)$ to $B(x_1, y_1)$ is at $(x_0 + r(x_1 - x_0), y_0 + r(y_1 - y_0))$.

7.3.3 Math Notes

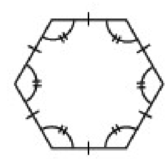
MATH NOTES

METHODS AND MEANINGS

Convex and Non-Convex Polygons

A **polygon** is defined as a two-dimensional closed figure made up of straight line segments connected end-to-end. These segments may not cross (intersect) at any other points.

A polygon is referred to as a **regular polygon** if it is equilateral (all sides have the same length) and equiangular (all interior angles have equal measure). For example, the hexagon shown at right is a regular hexagon because all sides have the same length and each interior angle has the same measure.



A polygon is called **convex** if each pair of interior points can be connected by a segment without leaving the interior of the polygon. See the example of convex and non-convex shapes in problem 8-4.

8.1.1 Math Notes

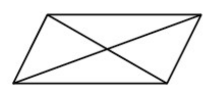
MATH NOTES

METHODS AND MEANINGS

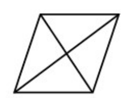
Special Quadrilateral Properties

In Chapter 7, you examined several special quadrilaterals and proved conjectures regarding many of their special properties. Review what you learned below.

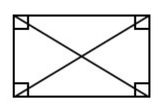
Parallelogram: Opposite sides of a parallelogram are congruent and parallel. Opposite angles are congruent. Also, since the diagonals create two pair of congruent triangles, the diagonals bisect each other.


Parallelogram

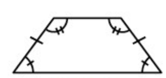
Rhombus: Since a rhombus is a parallelogram, it has all of the properties of a parallelogram. In addition, its diagonals are perpendicular bisectors that bisect the angles of the rhombus; the diagonals also create four congruent triangles.


Rhombus

Rectangle: Since a rectangle is a parallelogram, it has all of the properties of a parallelogram. In addition, its diagonals must be congruent.


Rectangle

Isosceles Trapezoid: The base angles (angles joined by a base) of an isosceles trapezoid are congruent.


Isosceles Trapezoid

METHODS AND MEANINGS

MATH NOTES

Interior and Exterior Angles of a Polygon

The properties of interior and exterior angles in polygons (which includes regular and non-regular polygons), where n represents the number of sides in the polygon (n-gon), can be summarized as follows:

- The sum of the measures of the interior angles of an n-gon is $180°(n-2)$.
- The sum of the measures of the exterior angles of an n-gon is always 360°.
- The measure of any interior angle plus its corresponding exterior angle is 180°.

In addition, for *regular* polygons:

- The measure of each interior angle in a regular n-gon is $\frac{180°(n-2)}{n}$.
- The measure of each exterior angle in a regular n-gon is $\frac{360°}{n}$.

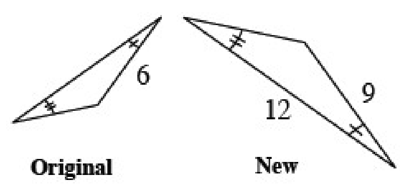
MATH NOTES

METHODS AND MEANINGS

Ratios of Similarity

Since Chapter 3, you have used the term **zoom factor** to refer to the ratio of corresponding dimensions of two similar figures. However, now that you will be using other ratios of similar figures (such as the ratio of the areas), this ratio needs a more descriptive name. From now on, this text will refer to the ratio of corresponding sides as the **linear scale factor**. The word "linear" is a reference to the fact that the ratio of the lengths of line segments is a comparison of a single dimension of the shapes. Often, this value is represented with the letter r, for ratio.

For example, notice that the two triangles at right are similar because of AA ~. Since the corresponding sides of the new and original shape are 9 and 6, it can be stated that $r = \frac{9}{6} = \frac{3}{2}$.



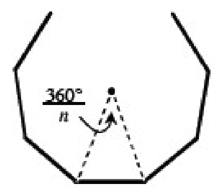
Original **New**

METHODS AND MEANINGS

MATH NOTES

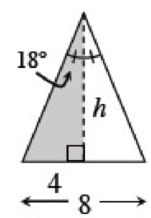
Area of a Regular Polygon

If a polygon is regular with n sides, it can be subdivided into n congruent isosceles triangles. One way to calculate the area of a regular polygon is to multiply the area of one isosceles triangle by n.



***n*-gon**

To find the area of the isosceles triangle, it is helpful to first find the measure of the polygon's central angle by dividing 360° by n. The height of the isosceles triangle divides the top vertex angle in half.



For example, suppose you want to find the area of a regular decagon with side length 8 units. The central angle is $\frac{360°}{10} = 36°$. Then the top angle of the shaded right triangle at right would be $36° \div 2 = 18°$.

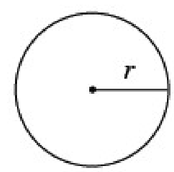
Use right triangle trigonometry to find the measurements of the right triangle, then calculate its area. For the shaded triangle above, $\tan 18° = \frac{4}{h}$ and $h \approx 12.311$. Use the height and the base to find the area of the isosceles triangle: $\frac{1}{2}(8)(12.311) \approx 49.242$ sq. units. Then the area of the regular decagon is approximately $10 \cdot 49.242 \approx 492.42$ sq. units.

METHODS AND MEANINGS

MATH NOTES

Circle Facts

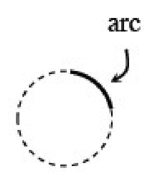
The area of a circle with radius $r = 1$ unit is π units^2. (Remember that $\pi \approx 3.1415926\ldots$)



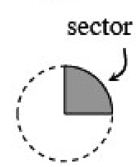
Area = πr^2
Circumference = $2\pi r = \pi d$

Since all circles are similar, their areas increase by a square of the linear scale (zoom) factor. That is, a circle with radius 6 has an area that is 36 times the area of a circle with radius 1. Thus, a circle with radius 6 has an area of 36π units^2, and a circle with radius r has **area** $A = \pi r^2$ units^2.

The **circumference** of a circle is its perimeter. It is the distance around a circle. The circumference of a circle with radius $r = 1$ unit is 2π units. Since the perimeter ratio is equal to the ratio of similarity, a circle with radius r has circumference $C = 2\pi r$ units. Since the diameter of a circle is twice its radius, another way to calculate the circumference is $C = \pi d$ units.



A part of a circle is called an **arc**. This is a set of points a fixed distance from a center and is defined by a central angle. Since a circle does not include its interior region, an arc is like the edge of a crust of a slice of pizza.

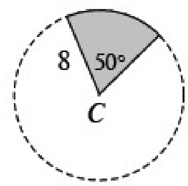


A region that resembles a slice of pizza is called a **sector**. It is formed by two radii of a central angle and the arc between their endpoints on the circle.

METHODS AND MEANINGS

MATH NOTES

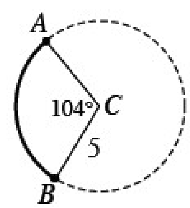
Arc Length and Area of a Sector



The ratio of the area of a sector to the area of a circle with the same radius equals the ratio of its central angle to 360°. For example, for the sector in circle C at right, the area of the entire circle is $\pi(8)^2 = 64\pi$ square units. Since the central angle is 50°, then the area of the sector can be found with the proportional equation:

$$\frac{50°}{360°} = \frac{\text{area of sector}}{64\pi}$$

Thus, the area of the sector is $\frac{50°}{360°}(64\pi) = \frac{80\pi}{9} \approx 27.93$ sq. units.



The length of an arc can be found using a similar process. The ratio of the length of an arc to the circumference of a circle with the same radius equals the ratio of its central angle to 360°. To find the length of $\overset{\frown}{AB}$ at right, first find the circumference of the entire circle, which is $2\pi(5) = 10\pi$ units. Then:

$$\frac{104°}{360°} = \frac{\text{arc length}}{10\pi}$$

Multiplying both sides of the equation by 10π, the arc length is $\frac{104°}{360°}(10\pi) = \frac{26\pi}{9} \approx 9.08$ units.

You may be surprised to learn that there are other units of measure (besides degrees) to measure an angle. A very special angle measure is called a **radian** and is defined as the measure of a central angle when the length of the arc equals the length of the radius. 1 radian = $\frac{360°}{2\pi} \approx 57.296°$.

MATH NOTES-Chapters 7 - 12

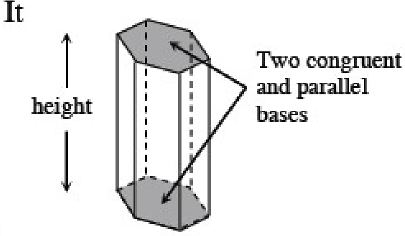
MATH NOTES

METHODS AND MEANINGS

Polyhedra and Prisms

A closed three-dimensional solid that has flat, polygonal faces is called a **polyhedron**. The plural of polyhedron is **polyhedra**. “Poly” is the Greek root for “many,” and “hedra” is the Greek root for “faces.”

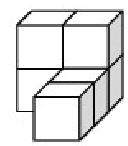
A **prism** is a special type of polyhedron. It must have two congruent, parallel **bases** that are polygons. Also, its **lateral faces** (the faces connecting the bases) are parallelograms formed by connecting the corresponding vertices of the two bases. Note that lateral faces may be any type of parallelogram, such as rectangles, rhombi, or squares.



MATH NOTES

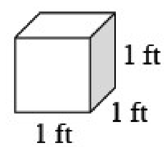
METHODS AND MEANINGS

Volume and Total Surface Area of a Solid



Volume measures the size of a three-dimensional space enclosed within an object. It is expressed as the number of $1 \times 1 \times 1$ cubes (or parts of cubes) that fit inside a solid.

For example, the solid shown above right has a volume of 6 cubic units.



Since volume reflects the number of cubes that fit within a solid, it is measured in **cubic units**. For example, if the dimensions of a solid are measured in feet, then the volume would be measured in cubic feet (a cube with dimensions $1' \times 1' \times 1'$).

On the other hand, the total **surface area** of a solid is the area of all of the external faces of the solid. For example, the total surface area of the solid above is 24 square units.

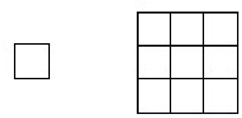
Cavalieri's Principle states that when the corresponding slices of two solids (with equal heights) have equal area, then the solids have equal volume. One way to think about Cavalieri's Principle is to think about how the volume of a stack of identical books would not change when you slide or twist some of them in the stack.

METHODS AND MEANINGS

MATH NOTES

The $r : r^2 : r^3$ Ratios of Similarity

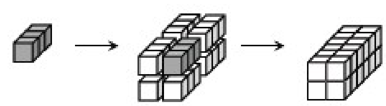
When a two-dimensional figure is enlarged proportionally, its perimeter and area also grow. If the linear scale factor is r, then the perimeter of the figure is enlarged by a factor of r while the area of the figure is enlarged by a factor of r^2. Examine what happens when the square at right is enlarged by a linear scale factor of 3.



$P = 4$ units, $A = 1$ units^2

$P = 4 \cdot \mathbf{3} = 12$ units, $A = 1 \cdot \mathbf{3}^2 = 9$ units^2

When a solid is enlarged proportionally, its surface area and volume also grow. If it is enlarged by a linear scale factor of r, then the surface area grows by a factor of r^2 and the volume grows by a factor of r^3. The example at right shows what happens to a solid when it is enlarged by a linear scale factor of 2.



Original solid SA = 14 units^2 V = 3 units^3

Width, height, and depth are doubled

Result: SA = 56 units^2 V = 24 units^3

Thus, if a solid is enlarged proportionally by a linear scale factor of r, then:

New edge length = $r \cdot$ (corresponding edge length of original solid)

New surface area = $r^2 \cdot$ (original surface area)

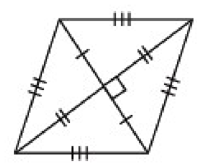
New volume = $r^3 \cdot$ (original volume)

METHODS AND MEANINGS

MATH NOTES

Rhombus Facts

Review what you have previously learned about a rhombus below.



A **rhombus** is a quadrilateral with four equal sides. All rhombi (the plural of rhombus) are parallelograms.

Starting with the definition of a rhombus above, there are several facts about rhombi that can be proved. For example, the diagonals of a rhombus are perpendicular bisectors of each other. That is, they intersect each other at their midpoints and form right angles at that point. Also, all four small triangles are congruent (as well as the larger triangles formed by any two of the smaller triangles). In addition, the diagonals of a rhombus bisect the opposite angles.

METHODS AND MEANINGS

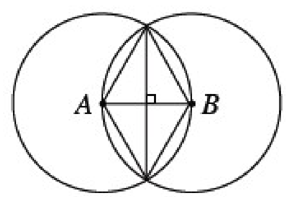
MATH NOTES

Constructing a Perpendicular Bisector

A perpendicular bisector of a given segment can be constructed using tracing paper or using a compass and a straightedge.

With tracing paper: To construct a perpendicular bisector with tracing paper, first copy the line segment onto the tracing paper. Then fold the tracing paper so that the endpoints coincide (so that they lie on top of each other). When the paper is unfolded, the resulting crease is the perpendicular bisector of the line segment.

With a compass and a straightedge: One way to construct a perpendicular bisector with a compass and a straightedge is to construct a circle at each endpoint of the line segment with a radius equal to the length of the line segment. Then use the straightedge to draw a line through the two points where the circles intersect. This line will be the perpendicular bisector of the line segment.

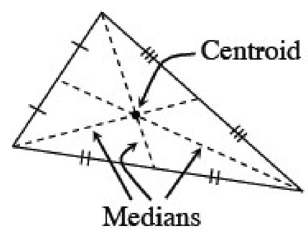


MATH NOTES

METHODS AND MEANINGS

Centroid and Medians of a Triangle

A line segment connecting a vertex of a triangle to the midpoint of the side opposite the vertex is called a **median**.



Since a triangle has three vertices, it has three medians. An example of a triangle with its three medians is provided at right.

The point at which the three medians intersect is called a **centroid**. The centroid is also the center of balance of a triangle.

Since the three medians intersect at a single point, this point is called a **point of concurrency**. You will learn about other points of concurrency in a later chapter.

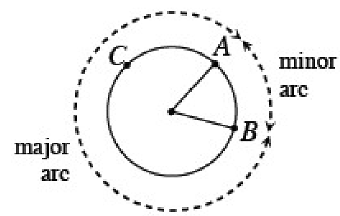
METHODS AND MEANINGS

MATH NOTES

Circle Vocabulary

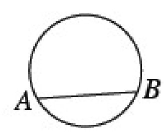
An **arc** is a part of a circle. Remember that a circle does not contain its interior. A bicycle tire is an example of a circle. The spokes and the space in between them are not part of the circle. The piece of tire between any two spokes of the bicycle wheel is an example of an arc.

Any two points on a circle create two arcs. When these arcs are not the same length, the larger arc is referred to as the **major arc**, while the smaller arc is referred to as the **minor arc**.



To name an arc, an arc symbol is drawn over the endpoints, such as $\overset{\frown}{AB}$. To refer to a major arc, a third point on the arc should be used to identify the arc clearly, such as $\overset{\frown}{ACB}$.

A **chord** is a line segment that has both endpoints on a circle. $\overline{AB}$ in the diagram at right is an example of a chord. When a chord passes through the center of the circle, it is called a **diameter**.



METHODS AND MEANINGS

MATH NOTES

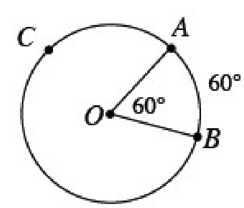
More Circle Vocabulary

The vertex of a **central angle** is at the center of a circle. An **inscribed angle** has its vertex on the circle with each side intersecting the circle at a different point.

One way to discuss an arc is to consider it as a fraction of 360°, that is, as a part of a full circle. When speaking about an arc using degrees, this is called the **arc measure**. The arc between the endpoints of the sides of a central angle has the same measure (in degrees) as its corresponding central angle.

When you want to know how *far* it is from one point to another as you travel along an arc, you call this the **arc length** and measure it in feet, miles, etc.

For example, point O is the center of $\odot O$ at right, and $\angle AOB$ is a central angle. The sides of the angle intersect the circle at points A and B, so $\angle AOB$ intercepts $\overset{\frown}{AB}$. In this case, the measure of $\overset{\frown}{AB}$ is 60°, while the measure of the major arc, $m\overset{\frown}{ACB}$, is 300° because the sum of the major and minor arcs is 360°. The length of $\overset{\frown}{AB}$ is $\frac{60}{360} = \frac{1}{6}$ of the circumference.

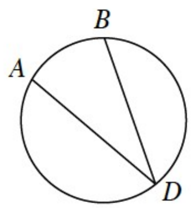


METHODS AND MEANINGS

MATH NOTES

Inscribed Angle Theorem

The measure of any inscribed angle is half of the measure of its intercepted arc. Likewise, any intercepted arc is twice the measure of any inscribed angles whose sides pass through the endpoints of the arc.

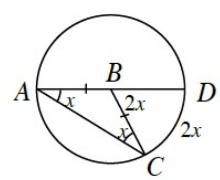


For example, in the diagram at right:

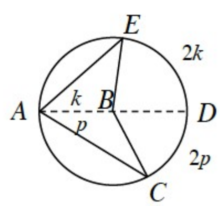
$$m\angle ADB = \tfrac{1}{2}\, m\widehat{AB} \text{ and } m\widehat{AB} = 2m\angle ADB$$

Proof:

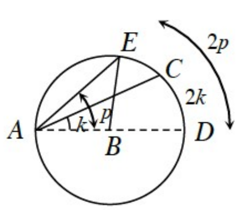
To prove this relationship, consider the relationship between an inscribed angle and its corresponding central angle. In problem 10-7, you used the isosceles triangle ΔABC to demonstrate that if one of the sides of the inscribed angle is a diameter of the circle, then the inscribed angle must be half of the measure of the corresponding central angle. Therefore, in the diagram at right, $m\angle DAC = \tfrac{1}{2}\, m\widehat{DC}$.



But what if the center of the circle instead lies in the interior of an inscribed angle, such as $\angle EAC$ shown at right? By extending $\overline{AB}$ to construct the diameter $\overline{AD}$, the work above shows that if $m\angle EAD = k$ then $m\widehat{ED} = 2k$ and if $m\angle DAC = p$, then $m\widehat{DC} = 2p$. Since $m\angle EAC = k + p$, then $m\widehat{EC} = 2k + 2p = 2(k + p) = 2m\angle EAC$.



The last possible case to consider is when the center lies outside of the inscribed angle, as shown at right. Again, constructing a diameter $\overline{AD}$ helps show that if $m\angle CAD = k$ then $m\widehat{CD} = 2k$ and if $m\angle EAD = p$, then $m\widehat{ED} = 2p$. Since $m\angle EAC = p - k$, then $m\widehat{EC} = 2p - 2k = 2(p - k) = 2m\angle EAC$.



Therefore, an arc is always twice the measure of any inscribed angle that intercepts it.

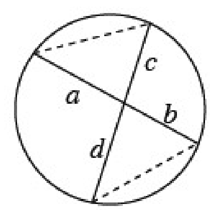
MATH NOTES

METHODS AND MEANINGS

Intersecting Chords

When two chords in a circle intersect, an interesting relationship between the lengths of the resulting segments occurs. If the ends of the chords are connected as shown in the diagram, similar triangles are formed (see problem 10-27). Then, since corresponding sides of similar triangles have a common ratio, $\frac{a}{d} = \frac{c}{b}$, and so

$$ab = cd.$$

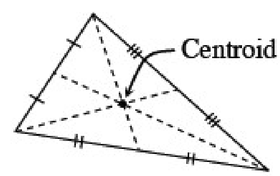


MATH NOTES

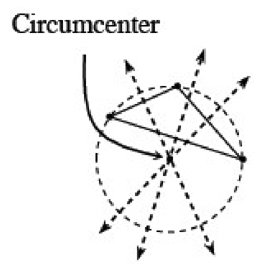
METHODS AND MEANINGS

Points of Concurrency

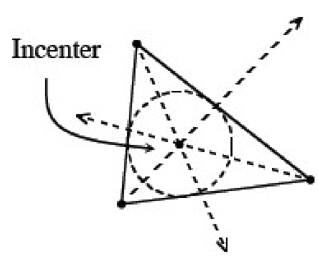
You learned that the **centroid** of a triangle is the point at which the three medians of a triangle intersect, as shown at right. When three lines intersect at a single point, that point is called a **point of concurrency**. Refer to the Math Notes box in Lesson 9.2.4.



A circle that **circumscribes** a triangle touches all three vertices of the triangle. The center of this circle is called the **circumcenter**. The circumcenter is another point of concurrency because it is located where the perpendicular bisectors of each side of a triangle meet. See the example at right.



A circle that **inscribes** a triangle touches all three sides of the triangle just once. The center of this circle is called the **incenter**. The incenter is yet another point of concurrency because it is located where the three angle bisectors of a triangle meet.



MATH NOTES

METHODS AND MEANINGS

Mutually Exclusive

Mutually exclusive events, also called **disjoint** events, can never both happen at the same time. When one of the events occurs, it means the other cannot possibly occur. If event B occurs, then you know that event A cannot occur: P(A and B) = 0 and the intersection of {A} and {B} contains no outcomes.

If events A and B are mutually exclusive, the occurrence of B tells you precisely about the probability of A occurring (A cannot occur). The probabilities of mutually exclusive events depend on each other. Mutually exclusive events are never independent (and thus always associated).

For example, suppose natural blondes occur in about 10% of the students at your school. Being naturally blonde and having naturally black hair are mutually exclusive – if one occurs, the other cannot possibly occur. If your friend tells you that a randomly selected person has naturally black hair, the probability they have naturally blonde hair is 0%. The probability of blonde has changed, knowing that the person has black hair. The events {blonde hair} and {black hair} are not independent.

METHODS AND MEANINGS

Conditional Probability and Independence

When you are calculating a probability, but have been given additional information about an event that has already occurred, you are calculating a **conditional probability**. For the conditional probability P(A given B), you know that event B has occurred, so event B becomes the sample space of all possible outcomes. P(A given B) is the fraction of event B's outcomes that also include event A, which is formally stated as the **Multiplication Rule**:

$$P(\text{A given B}) = \frac{P(\text{A and B})}{P(\text{B})}$$

Two events are **independent** when the outcome of one does not influence the outcome of the other. Two independent events could both occur, but knowing event B has occurred does not change the probability of event A occurring, thus P(A given B) = P(A). When events are not independent, you say that they are associated. That is, one event influences the other.

If you substitute the definition for independence, P(A given B) = P(A), into the Multiplication Rule and rearrange the result, you get an alternate definition for independence: If events A and B are independent, then P(A and B) = P(A) · P(B). The converse of this statement is also true: If P(A and B) = P(A) · P(B), then A and B are independent.

METHODS AND MEANINGS

MATH NOTES

Fundamental Principle of Counting

The **Fundamental Principle of Counting** is a method for counting the number of outcomes (the size of the sample space) of a probabilistic situation, often where the order of the outcomes matters. If event {A} has m outcomes, and event {B} has n outcomes after event {A} has occurred, then the event {A} followed by event {B} has $m \bullet n$ outcomes.

For a sequence of events, a tree diagram could be used to count the number of outcomes, but if the number of outcomes is large a **decision chart** is more useful.

For example, how many three-letter arrangements could be made by lining up any three blocks, chosen from a set of 26 alphabet blocks, if the first letter must be a vowel? There are three decisions (three blocks to be chosen), with 5 choices for the first letter (a vowel), 25 for the second, and 24 for the third. According to the Fundamental Principle of Counting, the total number of possibilities is:

$$\frac{5}{\text{1st decision}} \cdot \frac{25}{\text{2nd decision}} \cdot \frac{24}{\text{3rd decision}} = 3000 .$$

This decision chart is a way to represent a tree with 5 branches for the first alphabet block, followed by 25 branches for each of those branches; each of those 125 branches would then have 24 branches representing the possibilities for the third alphabet block.

MATH NOTES

METHODS AND MEANINGS

n! and Permutations

A **factorial** is shorthand for the product of a list of consecutive, descending whole numbers from the largest down to 1:

$$n! = n(n-1)(n-2)\ \ldots\ (3)(2)(1)$$

For example, 4 factorial or $4! = 4 \bullet 3 \bullet 2 \bullet 1 = 24$ and $6! = 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 720$.

A **permutation** is an arrangement of items in which the order of selection matters and items cannot be selected more than once. The number of permutations that can be made by selecting r items from a set of n items can be represented with tree diagrams or decision charts, or calculated ${}_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)...(n-r+1)$.

For example, eight people are running a race. In how many different ways can they come in first, second, and third? The result can be represented ${}_8P_3$, which means the number of ways to choose *and* arrange three different (not repeated) things from a set of eight.

$${}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = 8\cdot7\cdot6 = 336$$

METHODS AND MEANINGS

MATH NOTES

Combinations

When selecting committees, it matters who is selected but not the **order** of selection or any **arrangement** of the groups. Selections of committees, or of lists of groups without regard to the order within the group, are called **combinations**. Note that combinations do not include repeated elements.

For example: Eight people are eligible to receive $500 scholarships, but only three will be selected. How many different ways are there to select a group of three?

This is a problem of counting combinations. ${}_8C_3$ represents the number of ways to choose three from a set of eight. This is sometimes read as "eight *choose* three."

To compute the number of combinations, first calculate the number of permutations and then divide by the number of ways to arrange each permutation.

$${}_8C_3 = \frac{{}_8P_3}{3!} = \frac{8!}{5!3!} = 56$$

In general: Number of ways to choose $= \frac{\text{\# of ways to choose and arrange}}{\text{\# of ways to arrange}}$

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!r!}$$

MATH NOTES

METHODS AND MEANINGS

Definition of 0!

The use of the combinations formula when $r = n$ (when the number to be chosen is the same as the total number in the group) leads to a dilemma, as illustrated in the following example.

Suppose the Spirit Club has a total of three faithful members. Only one three-member governance committee is possible. If you apply the formula for combinations, you get ${}_3C_3 = \frac{{}_3P_3}{3!} = \frac{3!}{(3-3)!3!} = \frac{3!}{0!3!} = 1$. Does this make sense?

To resolve this question and make the formulas useful for all cases, mathematicians decided on this definition: $0! = 1$.

METHODS AND MEANINGS

MATH NOTES

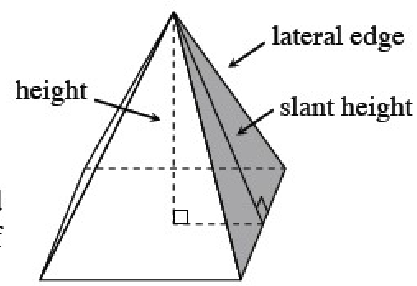
Pyramid Vocabulary

If a face of a pyramid (defined in problem 11-22) or prism is not a base, it is called a **lateral face**.

The **lateral surface area** of a pyramid or prism is the sum of the areas of all faces of the pyramid or prism, not including the base(s). The area of the exterior of the TransAmerica building that needs cleaning (from problem 11-23) is an example of lateral surface area, since the exterior of the base of the pyramid cannot be cleaned.

The **total surface area** of a pyramid or prism is the sum of the areas of all faces, including the bases.

Sometimes saying the word "height" for a pyramid can be confusing, since it could refer to the height of one of the triangular faces or it could refer to the overall height of the pyramid. Therefore, the height of each lateral face is called a **slant height** to distinguish it from the **height** of the pyramid itself. See the diagram at right.



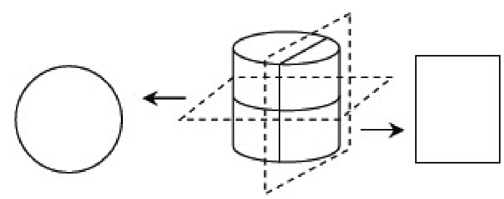
MATH NOTES

METHODS AND MEANINGS

Cross-Sections of Three-Dimensional Solids

The intersection of a three-dimensional solid and a plane is called a **cross-section** of the solid. The result is a two-dimensional diagram that represents the flat surface of a slice of the solid.

One way to visualize a cross-section is to imagine the solid sliced into thin slices like a ream of paper. Since a solid can be sliced in any direction and at any angle, you need to know the direction of the slice to find the correct cross-section.



A horizontal cross-section is a circle. | A cylinder | A vertical cross-section is a rectangle.

For example, the cylinder at right has several different cross-sections depending on the direction of the slice. When this cylinder is sliced vertically, the resulting cross-section is a rectangle, while the cross-section is a circle when the cylinder is sliced horizontally.

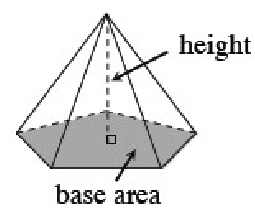
MATH NOTES

METHODS AND MEANINGS

Volume of a Pyramid

In general, the volume of a pyramid is one-third of the volume of the prism with the same base area and height. Thus:

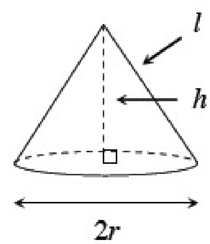
$$V = \frac{1}{3}(\text{base area})(\text{height})$$



MATH NOTES

METHODS AND MEANINGS

Volume and Lateral Surface of a Cone

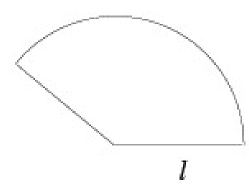


The general formula for the volume of a cone (defined in problem 11-54) is the same as the formula for the volume of a pyramid:

$$\text{Volume} = \tfrac{1}{3}(\text{Base Area})(\text{Height})$$

In the case of the cone, the Base Area = πr^2 where r is the length of the radius of the circular base. So if h is the height of the cone then the volume is:

$$V = \tfrac{1}{3}(\text{Base Area})(\text{Height}) = \tfrac{1}{3}\pi r^2 h\,.$$



To find the lateral surface area of a cone, imagine unrolling the lateral surface of the cone to create a sector. The radius of the sector would be the slant height, l, of the cone, and the arc length would be the circumference of the base of the cone, $2\pi r$.

Therefore, the area of the sector (the lateral surface area of the cone) is:

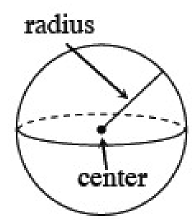
$$LA = \tfrac{2\pi r}{2\pi l}\pi l^2 = \pi r l$$

MATH NOTES

METHODS AND MEANINGS

Volume and Surface Area of a Sphere

A **sphere** is a three-dimensional solid formed by points that are equidistant from its center.



The **volume of a sphere** is twice the volume of a cone with the same radius and height. Since the volume of a cone with radius of length r and height $2r$ is $V = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3$, the volume of a sphere with radius of length r is:

$$V = \frac{4}{3}\pi r^3$$

The **surface area of a sphere** is four times the area of a circle with the same radius. Thus, the surface area of a sphere with radius of length r is:

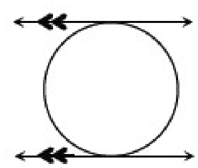
$$SA = 4\pi r^2$$

METHODS AND MEANINGS

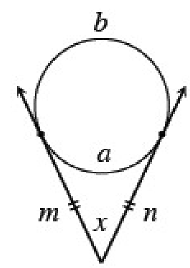
MATH NOTES

Intersecting Tangents

In some cases two lines that are tangent to the same circle do not intersect. When this happens, the tangent lines are parallel, as shown in the diagram at right. The arcs formed by the points of tangency are both 180°.



However, when the lines of tangency intersect outside the circle, some interesting relationships are formed. For example, the lengths m and n from the point of intersection to the points of tangency are equal.



The angle and arcs are related to the angle outside the circle as well. If x is the measure of the angle formed by the intersection of the tangents, a represents the measure of the minor arc, and b represents the measure of the major arc, then:

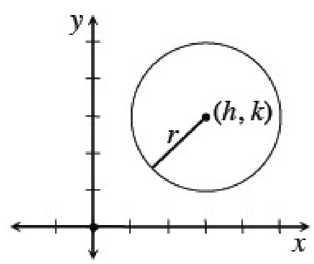
$$a = 180° - x \text{ and } b = 180° + x$$

MATH NOTES

METHODS AND MEANINGS

Equation of a Circle

The equation of a circle can be given in graphing form by the equation $(x-h)^2+(y-k)^2=r^2$, where (h, k) are the coordinates of the center of the circle, and r is the radius. Note that the equation of a circle does not describe a function, because there are two y-values for most x-values.



Alternatively, the equation of a circle can be written in general form as $ax^2+ay^2+bx+cy+d=0$ by multiplying the binomials. To rewrite the equation of a circle in general form to one in graphing form, the technique of completing the square can be used.

MATH NOTES

METHODS AND MEANINGS

Conic Sections

The cross-sections of a cone are also called **conic sections**. The shape of the cross-section depends on the angle of the slice. Three possible cross-sections of a cone (an ellipse, a parabola, and a hyperbola) are shown below. The other four conic sections are special cases of the first three (circle, line, point, and intersecting lines).